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Optimized interface conditions for domain decomposition methods in fluid dynamics

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SUMMARY

Interface conditions (IC) between subdomains have an important impact on the convergence rate of domain decomposition algorithms. We first recall the Schwarz method which is based on the use of Dirichlet conditions on the boundaries of the subdomains and overlapping subdomains. We explain how it is possible to replace them by more efficient ICs with normal and tangential derivatives so that overlapping is not necessary. It is possible to optimize the coefficients of the IC in order to achieve the best convergence rate. Results are given for the convection—diffusion equation. Then we consider the compressible Euler equations which form a system of equations. We present a new analysis of the use of interface conditions based on the flux splitting. We compute the convergence rate in the Fourier space. We find a dependence of their effectiveness on the Mach number M. For $M = \frac{1}{3}$, the convergence rate tends to zero as the wavenumber of the error goes to infinity. We stress the differences with the scalar equations. We present numerical results in agreement with the theoretical results. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: domain decomposition; interface condition; Euler; convection–diffusion

1. INTRODUCTION

We present results on the optimization of interface conditions for the convection–diffusion equation and the compressible Euler equations. The interface conditions have an important impact on the convergence of domain decomposition algorithms [1–8]. The question is well understood for scalar equations. We shall recall basic results for the scalar convection–diffusions equation. Systems of equations, such as the Euler equations, need further investigations. From our preliminary results presented here on this system of partial differential equations (PDEs), we shall stress the main and unexpected differences w.r.t the scalar case. In a joint paper by

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V. Dolean *et al*. in this volume, the Smith factorization is introduced in order to analyse and propose improved interface conditions.

The paper is organized as follows: in Section 2, the optimization procedure of the interface conditions for the scalar convection–diffusion is presented. In Section 3, the derivation of the 'classical' interface conditions for the Euler system is given. The convergence of the corresponding algorithm is analysed in Section 4. Differences with the scalar case are stressed in Section 5. Then we conclude in Section 6.

2. OPTIMIZED INTERFACE CONDITIONS FOR THE SCALAR CONVECTION–DIFFUSION EQUATION

2.1. The original Schwarz method (1870)

The first domain decomposition method was developed at the end of the 19th century by the mathematician H. Schwarz. His goal was to study the Laplace operator and not at all to develop numerical methods. At that time, the main tool for this purpose was Fourier analysis and more generally the use of special functions. Geometries of the domain were essentially restricted to simple geometries: rectangles and disks. His idea was to study the case of a domain that is the union of simple domains. For example, let $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 \neq \emptyset$. We want to solve (Figure 1)

$$-\Delta(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

H. Schwarz proposed the following algorithm (Multiplicative Schwarz Method, MSM):

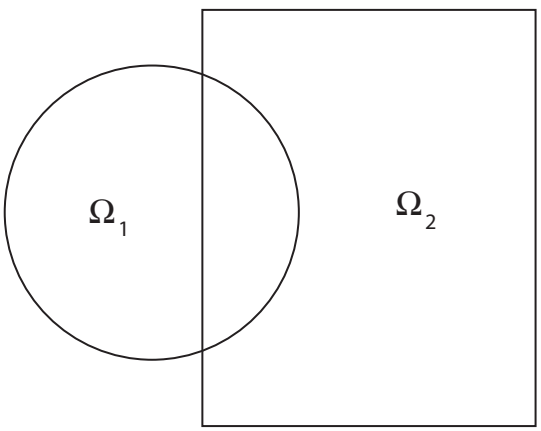

Figure 1. Overlapping domain decomposition.

Let (u_1^n, u_2^n) be an approximation to $(u_{|\Omega_1}, u_{|\Omega_2})$ at step n of the algorithm, (u_1^{n+1}, u_2^{n+1}) is defined by

$$
\begin{aligned}
&-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1, && -\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2\\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, && u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega\\
&u_1^{n+1} = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}, && u_2^{n+1} = u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}
\end{aligned}
$$

Problem in domain Ω_1 has to be solved before problem in domain Ω_2. This algorithm is sequential.

A slight modification of the algorithm is the additive Schwarz method (ASM)

$$
\begin{aligned}
&-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1, && -\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2\\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, && u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega\\
&u_1^{n+1} = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}, && u_2^{n+1} = u_1^n \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}
\end{aligned}
$$

Problems in domains Ω_1 and Ω_2 may be solved concurrently. The ASM is a parallel algorithm and is adapted to parallel computers. H. Schwarz proved the linear convergence of (u_1^n, u_2^n) to $(u_{|\Omega_1}, u_{|\Omega_2})$ as n tends to infinity.

The benefit of these algorithms is the saving in memory requirements. Indeed, if the problems are solved by direct methods, the cost of the storage is non-linear with respect to the number of unknowns. By dividing the original problem into smaller pieces the amount of storage can be significantly reduced.

2.2. Towards faster Schwarz methods

As far as CPU is concerned, the original algorithms ASM and MSM are very slow. Another weakness of the algorithms is the need of overlapping subdomains. In order to remedy to these drawbacks, it has been proposed [9] to replace the Dirichlet interface conditions on $\partial\Omega_i \backslash \partial\Omega$, $i = 1, 2$ by Robin interface conditions ($\partial_{n_i} + \alpha$, where n is the outward normal to subdomain Ω_i). For example, the modified ASM reads

$$
\begin{aligned}
-\Delta(u_1^{n+1}) &= f \quad \text{in } \Omega_1\\
u_1^{n+1} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega\\
\left(\frac{\partial}{\partial n_1} + \alpha\right)(u_1^{n+1}) &= \left(-\frac{\partial}{\partial n_2} + \alpha\right)(u_2^n) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}\\
-\Delta(u_2^{n+1}) &= f \quad \text{in } \Omega_2\\
u_2^{n+1} &= 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega\\
\left(\frac{\partial}{\partial n_2} + \alpha\right)(u_2^{n+1}) &= \left(-\frac{\partial}{\partial n_1} + \alpha\right)(u_1^n) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}
\end{aligned}
$$

Note that the normals n_1 and n_2 are opposite.

A good choice of the parameter α yields a much better convergence and the overlap between subdomains is optional. The boundary conditions imposed on $\partial\Omega_i \backslash \partial\Omega$, $i=1,2$ are called interface (or matching) conditions.

It is of course possible (and profitable) to consider even more general interface conditions of the form:

$$\text{on } \partial\Omega_1 \cap \overline{\Omega_2}: \quad (\partial_{n_1} + \alpha_1 + \beta_1 \partial_\tau - \gamma_1 \partial_{\tau^2}^2)(u_1^{n+1}) = (-\partial_{n_2} + \alpha_1 + \beta_1 \partial_\tau - \gamma_1 \partial_{\tau^2}^2)(u_2^n)$$

$$\text{on } \partial\Omega_2 \cap \overline{\Omega_1}: \quad (\partial_{n_2} + \alpha_2 + \beta_2 \partial_\tau - \gamma_2 \partial_{\tau^2}^2)(u_2^{n+1}) = (-\partial_{n_1} + \alpha_2 + \beta_2 \partial_\tau - \gamma_2 \partial_{\tau^2}^2)(u_1^n)$$

with $\alpha_i, \beta_i, \gamma_i$, $i=1,2$ being optimized for a fast convergence rate.

2.3. *Optimized interface conditions*

The convergence of the algorithm can be sharply analysed using Fourier transform. This will enable an optimization of the interface condition.

The model equation is the convection–diffusion equation:

$$\frac{\partial u}{\partial t} + \mathbf{a} \cdot \nabla u - \nu \Delta u = f \tag{1}$$

This equation is important in itself in engineering or environmental sciences for instance. It models the transport and diffusion of species (pollutant in air or water, electrons in semi-conductor devices, etc.) in a given flow (with velocity field $\mathbf{a}$). It is also a key ingredient in Navier–Stokes equations. An implicit scheme in time will demand at each time step the solution of

$$\mathscr{L}(u) \equiv \frac{u}{\Delta t} + \mathbf{a} \cdot \nabla u - \nu \Delta u = f \tag{2}$$

For the sake of simplicity, we consider the plane $\mathbb{R}^2$ decomposed into two half-planes $\Omega_1 =]-\infty, \delta[\times \mathbb{R}$ and $\Omega_2 =]0, \infty[\times \mathbb{R}$, $\delta \geqslant 0$.

Our results are based on the partial Fourier transform in the y direction denoted by $\hat{f}(x,k)$: let $f(x,y) : (l,L) \times \mathbb{R} \to \mathbb{R}$ then

$$\hat{f}(x,k) = \mathscr{F}_y(f)(x,k) := \int_{-\infty}^{\infty} \mathrm{e}^{-Iky} f(x,y)\, \mathrm{d}y$$

($I^2 = -1$) and the inverse Fourier transform of $\hat{f}(k)$ is given by

$$f(x,y) = \mathscr{F}_y^{-1}(\hat{f})(x,y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{Iky} \hat{f}(x,k)\, \mathrm{d}k$$

This tool is applied to the constant coefficient convection–diffusion equation, a_x and a_y are constants. The convergence rate can be computed explicitly. It shows that in order to have convergence, the interface condition must have the following form:

$$\partial_{n_i} + \frac{\mathbf{a} \cdot n_i}{2\nu} + \alpha + \beta \partial_\tau - \gamma \partial_{\tau^2}^2$$

with $\alpha, \gamma > 0$, β has the same sign as $\mathbf{a} \cdot \tau$ and where n_i is the outward normal to the subdomain and τ is an unit vector tangential to the interface. In other words, it is necessary to have

$\alpha_1 - \alpha_2 = \mathbf{a} \cdot n_1/2v$, $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$. In this case, the convergence rate of the error along the interface is given in the Fourier space by the following formula (see e.g. Reference [7]):

$$\rho(k;\alpha,\beta,\gamma) \equiv \left| \frac{\sqrt{(\mathbf{a}\cdot n)^2 + 4vc + 4I\mathbf{a}\cdot\tau vk + 4v^2k^2}}{2v} - (\alpha + I\beta k + \gamma k^2) \right| \times \mathrm{e}^{-\delta \frac{\sqrt{(\mathbf{a}\cdot n)^2 + 4vc + 4I\mathbf{a}\cdot\tau vk + 4v^2k^2}}{v}} \tag{3}$$

This formula deserves some comments:

- under the above assumption, the first term is smaller than 1 and the second term is smaller or equal to one. Therefore, convergence is ensured.
- the size of the overlap appears only in the second exponential term
- when the domains overlap $\delta > 0$, the second term is exponentially decreasing w.r.t. the wavenumber k. Thus, the high frequency part of the error has a very fast decay. For the low frequency part ($k \sim 0$), its decay depend on the value of $\sqrt{(\mathbf{a}\cdot n)^2 + 4vc}/2v$. When the flow is normal to the interface or when the time step is small, the exponential term is small and convergence is fast. However, such argument is not always true. For instance when large time steps are considered and when the flow is tangential to the interface, this term vanishes and it is not possible to rely on the overlap to ensure a fast convergence.

The non-overlapping (or very small overlap, $\delta = h$) case is currently in practice due to the need to avoid duplications of grid nodes or because of the use of non-matching grids. In this case, the convergence rate is given by the first term in (3). For any value of the parameters α, β and γ, this term tends to 1 as the wavenumber k goes to infinity. In a numerical setting, the wavenumber cannot become infinite but is bounded from above by π/h where h is the mesh size. This means that as the mesh size h goes to zero, the effective convergence rate which is the maximum of ρ over k

$$\rho(\alpha,\beta,\gamma) \equiv \max_{|k| \leqslant \pi/h} \rho(k;\alpha,\beta,\gamma)$$

deteriorates. For instance, if the parameters do not depend on h, it can be checked easily that $\rho \sim 1 - C^t h$ as h goes to zero. By optimizing the choice of the parameters α, β and γ, we have $\rho \sim 1 - C^t h^{1/3}$ as h goes to zero. The optimization problem to be solved is

Find $(\alpha_{\mathrm{opt}}, \beta_{\mathrm{opt}}, \gamma_{\mathrm{opt}})$ s.t.

$$\rho(\alpha_{\mathrm{opt}}, \beta_{\mathrm{opt}}, \gamma_{\mathrm{opt}}) = \min_{\alpha,\beta,\gamma} \rho(\alpha,\beta,\gamma) = \min_{\alpha,\beta,\gamma} \max_{|k| \leqslant \pi/h} \rho(k;\alpha,\beta,\gamma)$$

A detailed analysis is given in Reference [8]. Numerical results show the efficiency of this approach.

3. DERIVATION OF THE CLASSICAL INTERFACE CONDITIONS FOR THE EULER SYSTEM

We consider the time-dependent compressible Euler equations

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{F}(W) = 0$$
$$W = (\rho, \rho\mathbf{U}, E)^{\mathrm{T}}, \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^{\mathrm{T}}$$

discretized by an implicit scheme. At each time step, a non-linear system is solved

$$W^{n+1} + \Delta t \, \nabla \cdot \mathbf{F}(W^{n+1}) = W^n$$

by a Newton method. At each step k of the Newton method, the linearized Euler equations (where J is the Jacobian of $\nabla \cdot \mathbf{F}$) are solved

$$(Id + \Delta t J(W^{n+1,k}))\delta W^{n+1,k+1} = \cdots$$

by a Domain Decomposition Method.

3.1. The 1D case

The interval $[0, L]$ is decomposed into subintervals $([l_i, L_i])_{1 \leqslant i \leqslant N}$ with or without overlaps. The linearized system

$$A \frac{\partial W}{\partial x} + BW = G$$

$W = (\rho, u, p)$ is solved by a Schwarz-type algorithm

$$A \frac{\partial W_i^{k+1}}{\partial x} + BW_i^{k+1} = G \quad \text{in } (l_i, L_i)$$
$$C_i^+(W_i^{k+1}) = C_i^+(W_{i+1}^k) \quad \text{at } x = L_i$$
$$C_i^-(W_i^{k+1}) = C_i^-(W_{i-1}^k) \quad \text{at } x = l_i$$

where the matrices $C_i^{\pm}$ have to be chosen so that the subproblems are well posed and the algorithm has a fast convergence rate.

In the supersonic velocity case, all the interface conditions have to be imposed at inflow and no interface condition has to be imposed at outflow. C_i^- is an invertible 3×3 matrix. From that point of view of the algorithm, all the interface conditions yield the same algorithm

$$W_i^{k+1} = W_{i-1}^k \quad \text{at inflow}$$

and lead to a convergence in a number of steps equal to the number of subdomains.

In the subsonic velocity case, two interface conditions have to be imposed at inflow and one at outflow. They are given below. For a constant coefficient case (linearization around a

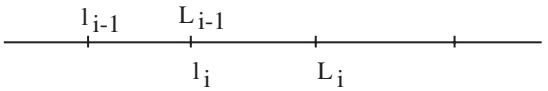

Figure 2. Decomposition of the interval into subdomains.

constant flow in the direction of positive x), optimal interface conditions can be designed by diagonalizing the system of equations.

$$A\frac{\partial W}{\partial x}+BW=G$$
$$\frac{\partial W}{\partial x}+A^{-1}BW=A^{-1}G$$

The matrix $A^{-1}B$ is diagonalized: $A^{-1}B=P^{-1}\ \mathrm{diag}(\lambda_j)_{j=1,\ldots,3}\,P$ with $\Re(\lambda_{1,2})>0$ and $\Re(\lambda_3)<0$.

Let $V:=PW$ and $H:=PA^{-1}G$, we have

$$\frac{\partial V_j}{\partial x}+\lambda_j V_j=H_j,\quad j=1,2,3$$

One optimal set of interface conditions for the interval (l_i,L_i) (see Figure 2) reads

$$V_{i,1}^{k+1}=V_{i-1,1}^{k}\quad\text{and}\quad V_{i,2}^{k+1}=V_{i-1,2}^{k}\quad\text{at } x=l_i$$
$$V_{i,3}^{k+1}=V_{i+1,3}^{k}\ \text{at } x=L_i$$

By using $V:=PW$, it is easy to express these relations in terms of the original unknowns. They yield convergence in a number of steps equal to the number of subdomains.

3.2. Two-dimensional case

The linearized system reads

$$A_x\frac{\partial W}{\partial x}+A_y\frac{\partial W}{\partial y}+BW=G$$

where $W=(\rho,u,v,p)$. The domain $[0,L]\times\mathbb{R}$ is decomposed into subintervals $([l_i,L_i]\times\mathbb{R})_{1\leqslant i\leqslant N}$ with or without overlaps.

The linearized system is solved by a Schwarz-type algorithm

$$A_x\frac{\partial W_i^{k+1}}{\partial x}+A_y\frac{\partial W_i^{k+1}}{\partial y}+BW_i^{k+1}=G\quad\text{in } (l_i,L_i)\times\mathbb{R}$$
$$C_i^{+}(W_i^{k+1})=C_i^{+}(W_{i+1}^{k})\quad\text{at } x=L_i$$
$$C_i^{-}(W_i^{k+1})=C_i^{-}(W_{i-1}^{k})\quad\text{at } x=l_i$$

where the matrices $C_i^{\pm}$ have to be chosen so that the subproblems are well posed and the algorithm has a fast convergence.

In the supersonic velocity case and as in the 1D case, any interface conditions that lead to a well-posed algorithm yield convergence in a number of steps equal to the number of subdomains.

In the subsonic velocity case, three interface conditions have to be imposed at inflow and one at outflow. For a constant coefficient case (linearization around a constant flow in the direction of positive x), optimal interface conditions can be designed by diagonalizing the system of equations in the Fourier space.

$$A_x \frac{\partial W}{\partial x} + A_y \frac{\partial W}{\partial y} + BW = G$$

$$\frac{\partial W}{\partial x} + A_x^{-1} A_y \frac{\partial W}{\partial y} + A_x^{-1} BW = A_x^{-1} G$$

We take the Fourier transform in the y direction of the above system ($W \to \tilde{W}(x,\xi)$, $G \to \tilde{G}(x,\xi)$).

$$\frac{\partial \tilde{W}}{\partial x} + A_x^{-1} A_y I\xi \tilde{W} + A_x^{-1} B\tilde{W} = A_x^{-1}\tilde{G}$$

where $I^2 = -1$. For each ξ, the matrix $A_x^{-1} A_y I\,\xi + A_x^{-1} B$ is diagonalized: $A_x^{-1} A_y I\,\xi + A_x^{-1} B = P^{-1}(\xi)\ \mathrm{diag}(\lambda_j(\xi))_{j=1,\ldots,4} P(\xi)$ with $\Re(\lambda_{1,2,3}(\xi)) > 0$ and $\Re(\lambda_4(\xi)) < 0$. Let $\tilde{V} := P(\xi)\tilde{W}$ and $\tilde{H} := PA_x^{-1}\tilde{G}$, we have

$$\frac{\partial \tilde{V}_j(x,\xi)}{\partial x} + \lambda_j(\xi)\tilde{V}_j(x,\xi) = \tilde{H}_j(x,\xi), \quad j = 1,2,3,4$$

One optimal set of interface conditions for the subdomain $(l_i, L_i) \times \mathbb{R}$ (see Figure 3) reads

$$\tilde{V}_{i,j}^{k+1}(l_i,\xi) = \tilde{V}_{i-1,j}^{k}(l_i,\xi), \quad j = 1,2,3$$

$$\tilde{V}_{i,4}^{k+1}(L_i,\xi) = \tilde{V}_{i+1,4}^{k}(L_i,\xi)$$

They yield convergence in a number of steps equal to the number of subdomains. By using $\tilde{V}(x,\xi) := P(\xi)\tilde{W}(x,\xi)$, the last one (e.g.) is equivalent to

$$\mathscr{F}^{-1}(P(\xi)\tilde{W}(L_i,\xi)_i^{k+1})_4 = \mathscr{F}^{-1}(P(\xi)\tilde{W}(L_i,\xi)_{i-1}^{k})_4$$

But, these interface conditions are pseudodifferential and difficult to use. These interface conditions are also exact absorbing boundary conditions. Classically, they can be approximated for $\xi = 0$. The approximation to the last interface condition reads

$$\mathscr{F}^{-1}(P(0)\tilde{W}(L_i,\xi)_i^{k+1})_4 = \mathscr{F}^{-1}(P(0)\tilde{W}(L_i,\xi)_{i-1}^{k})_4$$

or,

$$(P(0)W_i^{k+1})_4(L_i,y) = (P(0)W_{i-1}^{k})_4(L_i,y)$$

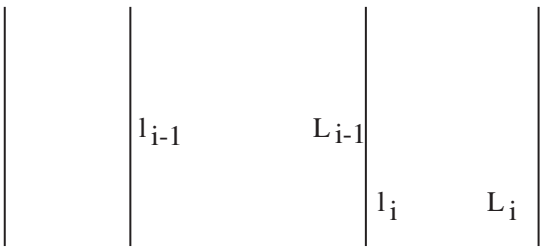

Figure 3. Decomposition of the plane into vertical strips.

The other ones read

$$(P(0)W_i^{k+1})_j(l_i, y) = (P(0)W_{i-1}^k)_j(l_i, y), \quad j = 1, 2, 3$$

In the next section, we analyse the convergence rate of a domain decomposition method based on these interface conditions.

4. ANALYSIS OF THE CLASSICAL INTERFACE CONDITIONS FOR THE EULER SYSTEM

In the case of a non-overlapping decomposition of the plane into two half-planes, the convergence rate of the error along the interface can be computed explicitly by a Fourier analysis similar to the one of the scalar case

$$\rho(\xi, M_n, M_t) = \left| \frac{R(\xi) - a}{(R(\xi) + a)^2} \cdot \frac{R(\xi)(1 - 3M_n) - a(1 + M_n)}{1 + M_n} \right|$$

where $R(\xi) = \sqrt{a^2 + \xi^2(1 - M_n^2)}$, $a = 1/c\Delta t + I\,\xi M_t$, M_n is the Mach number normal to the interface and M_t is the tangential Mach number (total Mach number $M = \sqrt{M_n^2 + M_t^2}$) and c is the sound speed.

Properties

1. $|\rho(\xi, M_n, M_t)| < 1$ and $\rho(0, M_n, M_t) = 0$
2. Special asymptotic behaviour at high wave numbers

$$|\rho(\infty, M_n, M_t)| = \sqrt{\left(\frac{1 - 3M_n}{1 + M_n}\right)^2 + \frac{8M_n M_t^2}{(1 + M_n)^2}}$$

so that $\rho(\infty,\ 1/3\ , 0) = 0$!!!!!

Property 1 resembles that of the scalar case. The first statement in Property 1 shows that there is convergence. The result is indeed general and has been proved for an arbitrary decomposition and a variable coefficient operator in Reference [10]. The second statement of Property 1 is to be expected. The optimal interface conditions have been approximated at $\xi = 0$.

5. COMPARISON BETWEEN THE SCALAR AND THE SYSTEM CASES

The main difference with respect to the scalar case comes from Property 2. In a numerical setting, the convergence rate can be estimated by

$$\rho(M_n, M_t) = \max_{|\xi| \leqslant \pi/h} \rho(\xi, M_n, M_t) \leqslant \max_{|\xi| \leqslant \infty} \rho(\xi, M_n, M_t) < 1$$

Property 2 therefore means that the iteration count is uniformly bounded as the mesh size h goes to zero. This is very different from the scalar case where the convergence rate deteriorates as h goes to zero, see Section 2.3. Moreover and surprisingly enough, for a velocity normal to the interface and a Mach number equals to $\frac{1}{3}$, the convergence rates tends to zero as ξ tends to infinity. Actually, the really meaningful quantity is the maximum over ξ of ρ as a function of the Mach number which is a bell curve (Figure 4) and which has a minimal value for M_n close to $\frac{1}{3}$.

In the scalar case, there is a monotone dependence of the convergence rate w.r.t. the normal component of the velocity at the interface.

These theoretical considerations agree qualitatively well with numerical tests performed at INRIA Sophia Antipolis: a flow in a tube divided into two half tubes is computed by a domain decomposition method using the classical interface conditions. This computation is

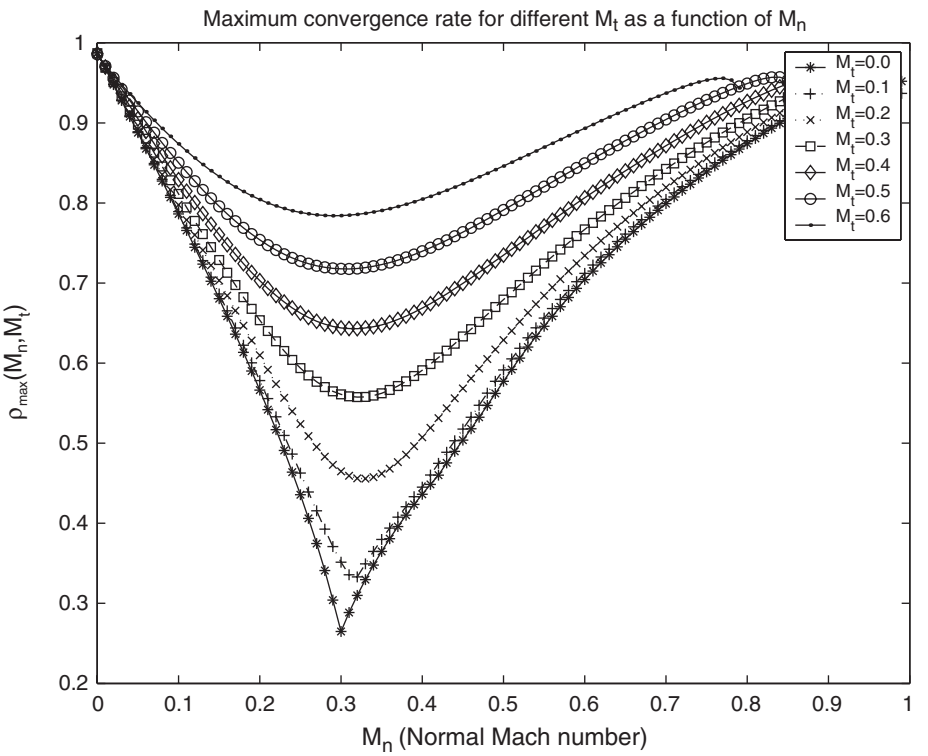

Figure 4. Convergence rate as a function of the Mach number.

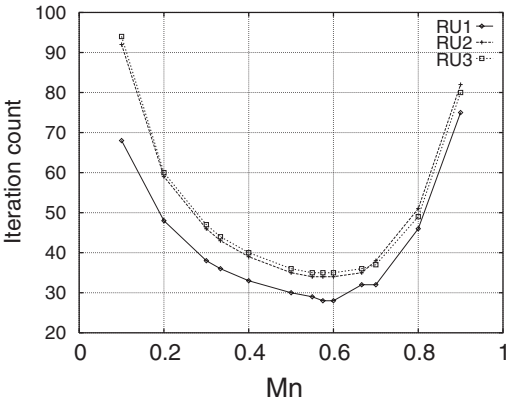

Figure 5. Number of iterations vs. the normal Mach number M_n for more and more refined meshes.

performed on three different meshes (RU1, RU2 and RU3) which are finer and finer (resp. 3740, 14 520 and 57 203 vertices). The number of iterations for the convergence of the domain decomposition method is plotted in Figure 5 as a function of the Mach number. We see that as the mesh size h is smaller, the number of iterations tends to a finite limit. Moreover, the curve is a 'bell' curve as predicted although the minimum is reached at $M_n \sim 0.6$ and is not close to $\frac{1}{3}$.

6. CONCLUSION AND PERSPECTIVES

These surprising results on domain decomposition methods for the system of compressible Euler equations are studied in a joint paper in this volume by V. Dolean *et al.* The analysis is based on the use of the Smith factorization of the matrices with polynomial entries. This enables the design of new interface conditions which are significantly better than the classical ones for Mach numbers close to 1.

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